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2003 J. Phys. A: Math. Gen. 36 8685

(http://iopscience.iop.org/0305-4470/36/32/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 8685–8701 PII: S0305-4470(03)59557-9

Numerical study of nonlinear equations with an infinite number of derivatives

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Received 12 February 2003, in final form 19 June 2003
Published 29 July 2003
Online at stacks.iop.org/JPhysA/36/8685

Abstract

We study equations with infinitely many derivatives. Equations of this type form a new class of equations in mathematical physics. These equations originally appeared in p-adic and later in fermionic string theories and their investigation is of much interest in mathematical physics and applications, in particular in cosmology. Differential equations with an infinite number of derivatives can be written as nonlinear integral equations. We perform a numerical investigation of the solutions of these equations. It is established that these equations have two different regimes of solutions: interpolating and periodic. The critical value of the parameter q separating these regimes is found to be $q_{\text{cr}}^2 \approx 1.37$. The convergence of the iterative procedure for these equations is proven.

PACS number: 02.30.−f

1. Introduction

Normally, in mathematical physics we consider equations with a finite number of derivatives. For such equations there are well-developed methods of solutions for various boundary problems; see, for example, [1]. Recently, in works on p-adic and then in real string theories a certain class of nonlinear equations, which involves an infinite number of derivatives, has begun to be explored [2–7]. Such equations form an important new class of equations in mathematical physics. New methods to study uniqueness and existence of solutions should be developed. It is not clear *a priori* how to pose boundary or initial value problems for these equations.

An example of new equations has the form

$$e^{a\Delta}\varphi = \varphi^k \tag{1}$$

where Δ is the Laplace (or D'Alamber) operator, a is a real parameter, and k is non-negative integer. This equation was originally studied in p-adic string theory. Soliton solutions to equation (1) were considered in [2, 3]. It is interesting to study equation (1) in the simplest case when $\varphi = \varphi(t)$ depends only on one real variable t [6, 7]. In this case, we can rewrite equation (1) in the form of an integral equation

$$(\mathcal{K}\varphi)(t) = \varphi(t)^k \tag{2}$$

where

$$e^{a\partial_t^2}\varphi(t) = (\mathcal{K}\varphi)(t) = \frac{1}{\sqrt{4a\pi}}\int_{-\infty}^{\infty} e^{-\frac{(t-t')^2}{4a}}\varphi(t')\,dt'. \tag{3}$$

In recent paper [6], a kink solution found in [2] was confirmed and also oscillatory solutions were found.

Equation (2) describes the dynamics of the scalar field (tachyon field) in p-adic string theory. It is expected that p-adic string theory possesses many non-perturbative features of open string field theory. The p-adic string theory for $p = 2$ is considered as a model for bosonic open string tachyon condensation and D-brane decay [7]. The p-adic string model for $p = 2$ has not only a perturbative vacuum but also a true (non-perturbative) vacuum in which there is no solution to the linearized equations of motion [6]. A similar situation is believed to occur in open string field theory where the open string excitations should disappear near the true vacuum and the D-brane should condense. Space-dependent solitonic lump solutions of equation (2) are interpreted as lower-dimensional D-branes. A time-dependent solution, which interpolates between two vacua (if it exists), is interpreted as D-brane decay and is called the rolling tachyon solution [7].

An interesting question is whether we can use p-adic models to study the condensation of the fermionic string tachyon in the $GSO(-)$ sector [8].

In this paper, we investigate the equations of the form (2) with $p = 3$ and more general equations. These equations describe the dynamics of the scalar field with the lowest mass square (tachyon field) in the fermionic string model [9]. Under a special approximation these equations are equivalent to a generalization of equation (2) that contains a parameter q^2. For a p-adic string $q^2 = 0$ and for a realistic fermionic string $q^2_{\text{string}} = -\left(4\ln\frac{4}{3\sqrt{3}}\right)^{-1} \approx 0.96$. We perform a numerical investigation of the solutions of these equations. It is established that these equations have two different regimes of solutions: interpolating and periodic, depending on the value of the parameter q^2. The critical value of the parameter q separating these regimes is found to be $q^2_{\text{cr}} \approx 1.37$. Our result, that $q^2_{\text{string}} < q^2_{\text{cr}}$, is important since it shows that p-adic string theory for $p = 3$ in fact has common features with the realistic fermionic string model. The convergence of the iterative procedure for these equations is proven. To construct a numerical algorithm we essentially used object-oriented design; see section 3.3 for further details.

This paper is organized as follows. In the next section we study the equation taken from the p-adic string model, and we prove the convergence of the iterative procedure for this equation. In the next two sections, we study equations describing the scalar field (tachyon field) in the fermionic string. Equations for the fermionic string generalize the equations for the p-adic string. Finally, in the last section we provide physical details including actions for which corresponding equations of motion form the subject of this paper.

2. Scalar field dynamics in the p-adic string model

In this section we study the following nonlinear integral equation

$$(\mathcal{K}\varphi)(t) = \varphi(t)^p \tag{4}$$

where the integral operator $\mathcal{K}$ is defined by

$$(\mathcal{K}\varphi)(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(t-t')^2} \varphi(t')\,\mathrm{d}t'. \tag{5}$$

Originally, p on the right-hand side (rhs) of equation (4) is a prime number, although here it is not important. This equation has several physical applications; basically it describes dynamics of homogeneous scalar field configurations in the p-adic string model [2–4]. We describe the physics behind this in more detail in section 5. Equation (4) for $p = 3$ was studied in [2]. It was numerically shown that it has a solution which goes to ∓ 1 on infinities. In this section we prove the convergence of the iterative procedure used in [2] to construct a numerical solution of equation (4). In [6], it was shown that equation (4) does not have monotonic solutions for even p. In the text below we discuss only the case $p = 3$ since it is the most illustrative for the p-adic string model and provides an approximation for the fermionic string model.

We consider equation (4) for the case $p = 3$

$$(\mathcal{K}\varphi)(t) = \varphi(t)^3. \tag{6}$$

We are searching for the solution which has constant asymptotic behaviour on infinities. For the case of the constant field, equation (6) takes the form

$$\varphi_0 = \varphi_0^3. \tag{7}$$

It has three solutions:

$$\varphi_0^{(1)} = 1 \qquad \varphi_0^{(2)} = 0 \qquad \varphi_0^{(3)} = -1. \tag{8}$$

We are interested in the odd solution of equation (6) which goes to ∓ 1 on infinities, i.e.

$$\lim_{t\to\pm\infty} \varphi(t) = \mp 1 \tag{9}$$

and

$$\varphi(t) = -\varphi(-t). \tag{10}$$

2.1. *Construction of a solution using the iterative procedure*

To construct a solution of the integral equation (6) with the properties (9)–(10) we can use the following iterative procedure [2, 12]

$$\varphi_{n+1} = (\mathcal{K}\varphi_n)^{1/3} \tag{11}$$

where zero approximation is taken as

$$\varphi_0(t) = -\varepsilon(t) \tag{12}$$

and where $\varepsilon(t)$ is a step function defined by

$$\varepsilon(t) = \begin{cases} -1 & \text{for} \quad t < 0 \\ 0 & \text{when} \quad t = 0 \\ 1 & \text{for} \quad t > 0. \end{cases} \tag{13}$$

Note that on the rhs of equation (12) the expression of the form $a^{1/3}$ denotes an arithmetic cubic root of a which is well defined for negative arguments as

$$a^{1/3} = \begin{cases} a^{1/3} & a \geqslant 0 \\ -|a|^{1/3} & a < 0. \end{cases} \tag{14}$$

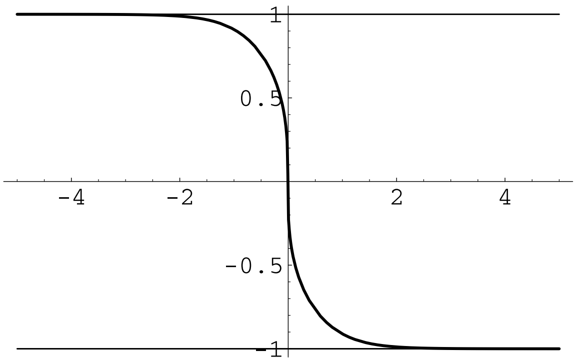

Figure 1. The results of the iterative procedure (11)–(12) for a large number of steps ($\sim 10^5$).

The results of the iterative procedure (11)–(12) are presented in figure 1.

In the next section we will prove the convergence of the iterative procedure (11)–(12) that supports the results of numerical computations in [2].

The first approximation $\varphi_1(t)$ can be computed analytically and is given by the arithmetical cubic root of the error function. Indeed

$$\varphi_1^3(t) = (\mathcal{K}\varphi_0)(t) = -\mathrm{erf}(t) \tag{15}$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,\mathrm{d}t.$$

In order to prove the convergence of iterative procedure (11) it is convenient to take into account the parity property (10). This allows us to rewrite equation (6) on the semi-axis

$$(\mathcal{K}_-\varphi)(t) = \varphi(t)^3 \qquad t < 0 \tag{16}$$

where the integral operator $\mathcal{K}_-$ is defined by

$$(\mathcal{K}_-\varphi)(t) = \int_{-\infty}^0 K_-(t,t')\varphi(t')\,\mathrm{d}t' \tag{17}$$

and where the kernel $K_-(t,t')$ is given by

$$K_-(t,t') = \frac{1}{\sqrt{\pi}}\left[\mathrm{e}^{-(t-t')^2} - \mathrm{e}^{-(t+t')^2}\right]. \tag{18}$$

Let us note that there is the following positiveness property of the kernel

$$K_-(t,t') > 0 \qquad \text{for all} \quad t < 0 \quad t' < 0 \tag{19}$$

and

$$K_-(t,0) = K_-(0,t') = 0$$

(see figure 2).

2.2. *Convergence of the iterative procedure*

Here we describe the construction of the solution for equation (16) on the semi-axis ($t < 0$). We consider the following iterative procedure

$$\varphi_{n+1} = (\mathcal{K}_-\varphi_n)^{1/3}. \tag{20}$$

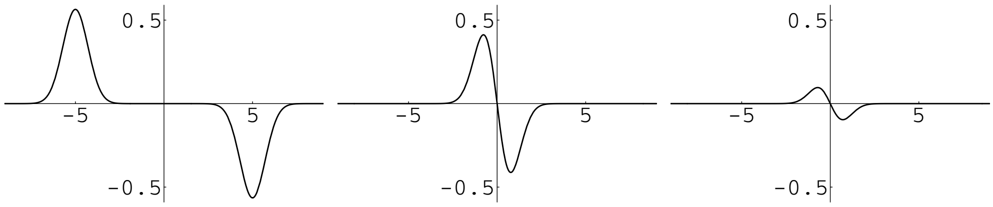

Figure 2. The kernel $K_-(b,t)$ as a function of t for $b = -5$, -0.5 and -0.1. It is positive for $t < 0$ and $b < 0$.

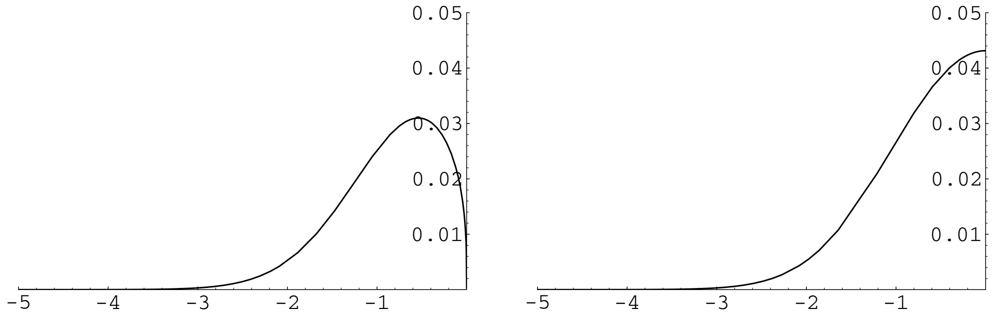

Figure 3. Plots of $\varphi_1(t) - \varphi_2(t)$ and $\Delta(t)$ illustrate that $\Delta(t) < 0.05$.

Let us note that the second iteration

$$\varphi_2 = (\mathcal{K}_-\varphi_1)^{1/3} \tag{21}$$

can be represented in the form

$$\varphi_2(t) = \varphi_1(t)(1 - \Delta(t)) \tag{22}$$

where $\Delta(t)$ is defined by

$$\Delta(t) = \frac{\varphi_1(t) - \varphi_2(t)}{\varphi_1(t)}. \tag{23}$$

In figure 3 the difference $(\varphi_1(t) - \varphi_2(t))$ and $\Delta(t)$ are shown for negative t. From equations (15), (21) and (23) it follows that (see figure 3)

$$\Delta(t) < \Delta_{\max} = 0.05 \tag{24}$$

thus

$$\varphi_1(x) > \varphi_2(x) > \varphi_1(x)(1 - \Delta_{\max}). \tag{25}$$

Since we have the positiveness property (19), we can integrate the inequality (25) with the kernel $K_-(t,t')$

$$\int_{-\infty}^{0} K_-(y,x)\varphi_1(x)\,\mathrm{d}x \geqslant \int_{-\infty}^{0} K_-(y,x)\varphi_2(x)\,\mathrm{d}x \geqslant (1 - \Delta_{\max}) \int_{-\infty}^{0} K_-(y,x)\varphi_1(x)\,\mathrm{d}x. \tag{26}$$

Inequality (26) leads us to

$$\varphi_2^3(y) \geqslant \varphi_3^3(y) \geqslant \varphi_2^3(y)(1 - \Delta_{\max}). \tag{27}$$

Now taking the arithmetical cubic root we obtain

$$\varphi_2(y) \geqslant \varphi_3(y) \geqslant \varphi_2(y)(1-\Delta_{\max})^{1/3} \tag{28}$$

and moreover

$$\varphi_1(y) \geqslant \varphi_2(y) \geqslant \varphi_3(y) \geqslant \varphi_2(y)(1-\Delta_{\max})^{1/3} \geqslant \varphi_1(y)(1-\Delta_{\max})^{1+1/3}. \tag{29}$$

Analogously we have

$$\varphi_1(y) \geqslant \varphi_n(y) \geqslant \varphi_1(y)(1-\Delta_{\max})^{1+1/3+(1/3)^2+\cdots+(1/3)^{n-2}} = \varphi_1(y)(1-\Delta_{\max})^{\frac{3}{2}-\frac{1}{2}(\frac{1}{3})^{n-2}} \tag{30}$$

i.e.

$$\varphi_1(y) \geqslant \varphi_n(y) \geqslant \varphi_1(y)(1-\Delta_{\max})^{\frac{3}{2}-\frac{1}{2}(\frac{1}{3})^{n-2}}. \tag{31}$$

From equation (31) it follows that $\varphi_n(y)$ is uniformly bounded on the whole negative semi-axis.

Moreover, besides equations (27) and (28) we have

$$(1-\Delta_{\max})^{1/3}\varphi_3^3(y) \leqslant \varphi_4^3(y) \leqslant \varphi_3^3(y) \tag{32}$$

thus

$$(1-\Delta_{\max})^{1/9}\varphi_3(y) \leqslant \varphi_4(y) \leqslant \varphi_3(y). \tag{33}$$

Analogously we have

$$(1-\Delta_{\max})^{1/3^{n-1}}\varphi_n(y) \leqslant \varphi_{n+1}(y) \leqslant \varphi_n(y). \tag{34}$$

Hence

$$|\varphi_n(y) - \varphi_{n+1}(y)| \leqslant \varphi_n(y)\left(1-(1-\Delta_{\max})^{\frac{1}{3^{n-1}}}\right). \tag{35}$$

Finally, taking into account that $\varphi_n(y)$ is uniformly bounded we obtain

$$|\varphi_n(y) - \varphi_{n+1}(y)| < \frac{C}{3^n} \tag{36}$$

where C is constant.

From equation (36) follows the uniform convergence of $\varphi_n(y)$ on the semi-axis, i.e. uniform convergence of the iterative procedure on the semi-axis.

3. Scalar field dynamics in fermionic string model I

3.1. Integro-differential equation

In this section we consider the following integro-differential equation

$$\left(-q^2\partial_t^2 + 1\right)(\mathcal{K}\varphi)(t) = \varphi(t)^3 \tag{37}$$

where the integral operator $\mathcal{K}$ is defined by equation (5) and q is a parameter. This equation describes dynamics of the homogeneous scalar field with the lowest mass square (tachyon) in the fermionic string model in some approximation; see section 5 for more physical details. Equation (37) transforms to the p-adic equation (6) in the case $q = 0$. Although the value of the parameter q for the fermionic string model is given by

$$q_{\text{string}}^2 = -\frac{1}{4\ln\frac{4}{3\sqrt{3}}} \approx 0.96 \tag{38}$$

we consider equation (37) for various values of parameter q.

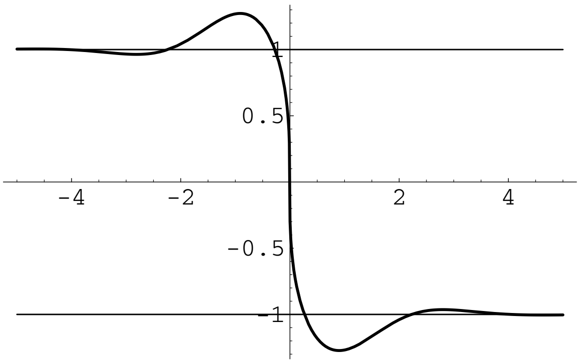

Figure 4. Results of the iterative procedure (41)–(42) for $q = q_{\mathrm{string}}$.

3.2. Fully integral form and iterative procedure

Equation (37) can be written in fully integral form

$$(\mathcal{K}_q\varphi)(t) = \varphi(t)^3 \tag{39}$$

where the kernel of integral operator $\mathcal{K}_q$ is obtained by differentiating the kernel of equation (5), namely by applying the operator $\left(-q^2\partial_t^2 + 1\right)$ to $\exp[-(t-t')^2]$

$$(\mathcal{K}_q\varphi)(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-(t-t')^2}[1 + 2q^2(1 - 2(t-t')^2)]\varphi(t')\,\mathrm{d}t'. \tag{40}$$

As in the previous section we are interested in the odd solution of equation (39) which goes to ∓ 1 on infinities, i.e. the solution of equation (39) with the properties (9)–(10).

To construct the solution we use the following iterative procedure

$$\varphi_{n+1} = (\mathcal{K}_q\varphi_n)^{1/3} \tag{41}$$

where zero approximation is taken as

$$\varphi_0(t) = -\varepsilon(t) \tag{42}$$

and $\varepsilon(t)$ is a step function defined by equation (13). In the previous section we proved that this iterative procedure converges to a solution for the case $q = 0$.

3.3. Numerical results

We use the iterative procedure (41)–(42) to perform a numerical investigation of the solution properties for various values of parameter q.

It was found that there exists a critical value of the parameter q^2: $q_{\mathrm{cr}}^2 \approx 1.37$. This is the maximum value of q for which there exist interpolating solutions. For $q^2 < q_{\mathrm{cr}}^2$ there were numerically found solutions with asymptotic behaviour (9) which oscillate along $-\varepsilon(t)$ with exponentially decreasing amplitude. Figure 4 demonstrates a numerical limit of the iterative procedure (41)–(42) for $q = q_{\mathrm{string}}$.

While increasing the value of q above the critical value q_{cr} there appears a swing and the solution of equation (39) becomes periodic (see figure 5). Let us note that this swing for $q^2 \simeq 1.37 \pm 0.01$ appears on quite large step numbers (values of n in equation (41)). In figure 6 the result of the iterative procedure for $q^2 = 1.38$ for various step numbers are shown.

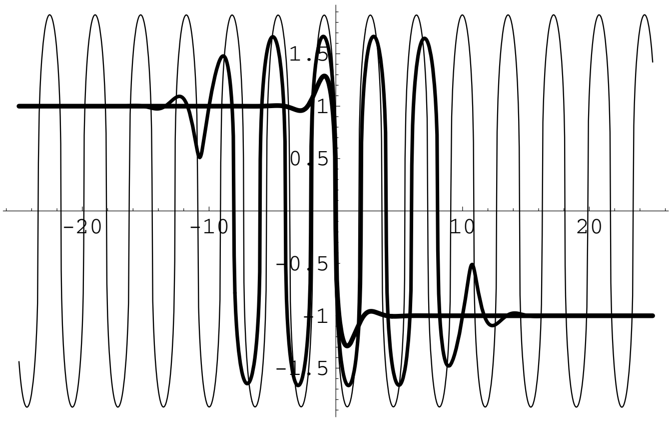

Figure 5. Appearance of the swing of the solution with the increase of q: $q^2 = 1$ (thick line), no swing, interpolating regime; $q^2 = 1.4$ (medium-thick line), *appearance* of the swing, oscillatory regime; $q^2 = 1.8$ (thin line), oscillatory regime.

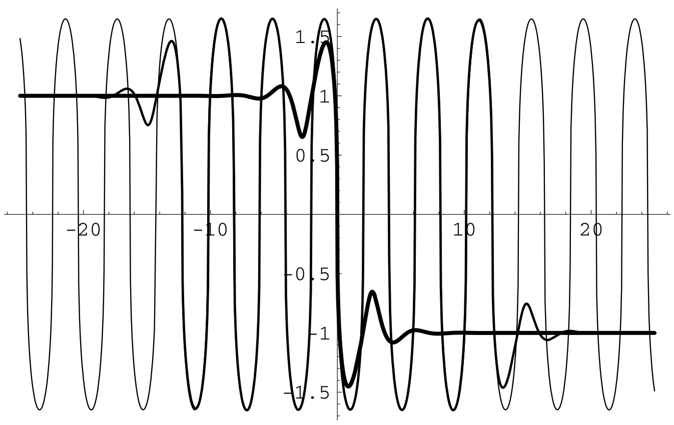

Figure 6. The transformation to the periodic regime for $q^2 = 1.38$ appears on quite large step numbers: 100th step (thick line); 250th step (medium-thick line); 500th step (thin line).

A numerical investigation of the iterative procedure for very high step numbers ($\sim 10^5$) has shown that transformation to the periodic regime does not appear for $q < q_{\mathrm{cr}}$. Although, for values of q slightly above the critical value q_{cr} the swing, i.e. a transformation to the periodic regime, appears for much smaller step numbers ($\sim 10^2$). This shows that there is a principle difference between solutions for q below and above q_{cr}. This fact is discussed in more detail in the next section.

Here we have used equation (42) as a zero approximation φ_0 which is not continuous at the point $t = 0$. To understand that this discontinuity does not affect the results of the iterative procedure, we tried several smooth continuous zero approximations which have asymptotic behaviour (9). In particular, we tried $-\frac{2}{\pi}\arctan(t)$ and $-\mathrm{erf}(t)$. The results of the iterative procedure with these zero approximations were for large step numbers, the same as with equation (42).

For numerical computation we used the following approximation for $\mathcal{K}$

$$(\mathcal{K}\varphi)(t) \cong \frac{1}{\sqrt{\pi}} \int_{t-\Delta}^{t+\Delta} \mathrm{e}^{-(t-t')^2} \varphi(t')\, \mathrm{d}t'. \tag{43}$$

Here Δ was automatically adjusted, namely $\Delta \simeq 10$. This approximation led us to the algorithm with linear complexity that allowed us to compute $\sim 10^5$ iterations on a 64-bit parallel Sun Ultra-SPARC machine. The entire algorithm was written in C++.

To construct the numerical algorithm we essentially used object-oriented design [10, 11]. This allowed us to develop a general algorithm which solves some class of integral equations; in particular, it was used to solve a system of nonlinear integral equations discussed in section 4.

3.4. Two regimes of the solution

In this section we are interested in the mechanism which forms the numerically found exponentially decreasing oscillations and the presence of q_{cr}. The basic idea is to present a solution with non-zero q as a deviation along the solution with $q = 0$. We write the linear equation for this deviation for large values of t.

Let us write the solution of equation (37) as a sum

$$\varphi(t) = \phi_0(t) + \chi(t) \tag{44}$$

where $\phi_0(t)$ denotes the solution of equation (4). Substituting equation (44) into equation (37) and leaving only linear terms in χ we obtain

$$\left(-q^2\partial_t^2 + 1\right)\mathcal{K}(\phi_0 + \chi) = \phi_0^3 + 3\phi_0^2\chi. \tag{45}$$

Now using the fact that ϕ_0 satisfies equation (4) we obtain the linear integro-differential equation on $\chi(t)$

$$\left(-q^2\partial_t^2 + 1\right)\mathcal{K}\chi = 3\phi_0^2\chi + q^2\partial_t^2\mathcal{K}\phi_0. \tag{46}$$

From equation (43) we see that, in $\mathcal{K}$, integration can be taken in finite limits with good precision. This gives us the ability to write a large t approximation of equation (46)

$$\left(-q^2\partial_t^2 + 1\right)\mathcal{K}\chi = 3\chi. \tag{47}$$

Here we used the fact that $\phi_0(t) \simeq 1$ and its derivatives are zero for large t. Representing $\chi(t)$ as

$$\chi(t) = \mathrm{e}^{\mathrm{i}\Omega t}$$

and substituting it into equation (47) we obtain the following characteristic equation

$$(q^2\Omega^2 + 1)\,\mathrm{e}^{-\frac{1}{4}\Omega^2} = 3. \tag{48}$$

We consider equation (48) as an equation for the complex variable Ω with parameter q^2. There is a minimum value of q^2 $\left(q_0^2 \approx 1.77\right)$ for which Ω is real. For $q^2 < q_0^2$ it has solutions with non-zero imaginary parts which give the oscillatory regime with exponentially decreasing amplitude. For $q^2 > q_0^2$ the solutions are real. As mentioned in the previous section, numerical computations give the critical value $q_{\mathrm{cr}}^2 \approx 1.37$ which is smaller than the value found above: $q_0^2 \approx 1.77$. This probably means that we cannot neglect the difference of ϕ_0 from 1 in the rhs of equation (46) or even that we have to take into account nonlinear terms in χ in equation (45). Nevertheless, the method discussed in this section provides a good qualitative explanation to the behaviour of solutions for small values of q.

3.5. *Periodic solutions for large q*

In this section we investigate the asymptotic behaviour of the solution for large q. First, let us introduce a new function $\chi(t)$ defined by

$$\chi(t) = \varphi(qt). \tag{49}$$

In terms of $\chi(t)$ equation (37) is rewritten as

$$\left(-\partial_t^2 + 1\right)(\mathcal{P}_q \chi)(t) = \chi(t)^3 \tag{50}$$

where the integral operator $\mathcal{P}_q$ is defined by

$$(\mathcal{P}_q \chi)(t) = \frac{q}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-q^2(t-t')^2} \chi(t')\,\mathrm{d}t'. \tag{51}$$

Noting that the solution of equation (50) depends on parameter q, let us write it as a power series in $1/q^2$. Introducing a change of variables $t' \to \tau = t + t'/q$ in the rhs of equation (51) we obtain

$$\frac{q}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-q^2\tau^2} \chi(t+\tau)\,\mathrm{d}\tau.$$

Now changing variables again $\tau \to \sigma = q\tau$ we get

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\sigma^2} \chi\left(t + \frac{\sigma}{q}\right) \mathrm{d}\sigma.$$

Finally, expanding $\chi(t + \sigma/q)$ in Taylor series with respect to σ we obtain

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\sigma^2} \left(\chi(t) + \frac{\sigma}{q}\chi'(t) + \frac{\sigma^2}{q^2}\chi''(t) + \cdots\right) \mathrm{d}\sigma.$$

All terms that are the odd powers of σ vanish after integration. We obtain

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 1\right)\left(1 + \frac{1}{4q^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \cdots\right)\chi = \chi^3$$

or

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 1\right) \mathrm{e}^{\frac{1}{4q^2}\partial_t^2} \chi = \chi^3 \tag{52}$$

where $\exp\left(1/4q^2\partial_t^2\right)$ is understood as a formal expansion

$$\mathrm{e}^{\frac{1}{4q^2}\partial_t^2} = \sum_{n=0}^{\infty} \left(\frac{1}{4q^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)^n.$$

Let us note that equation (52) can be directly obtained from the differential form of the equation; see section 5.

In a limit $q \to \infty$, equation (52) becomes an equation for the anharmonic oscillator. This explains numerically found periodic solutions for large q.

4. Scalar field dynamics in fermionic string model II

4.1. *System of nonlinear integral equations*

In this section we study the following system of integral equations

$$(\mathcal{K}\sigma)(t) = \varphi(t)^2 \tag{53}$$

$$\left(-q^2\partial_t^2+1\right)(\mathcal{K}\varphi)(t)=\sigma(t)\varphi(t) \tag{54}$$

where the integral operator $\mathcal{K}$ is defined by equation (5) and q is a parameter. Physically we are interested in solutions which are finite on the whole axis and $\varphi(t)$ is an odd function interpolating between ∓ 1. This system of equations describes the dynamics of the homogeneous scalar field with the lowest mass square (tachyon) in the fermionic string model in the first non-trivial approximation; see section 5 for more physical details.

We are interested in the solutions with the following properties. The function $\varphi(t)$ is odd and goes to ∓ 1 on infinities, i.e. satisfying equations (9)–(10), and the function $\sigma(t)$ is even and goes to 1 on infinities, i.e.

$$\sigma(t)=\sigma(-t) \qquad \sigma(\mp\infty)=1. \tag{55}$$

4.2. Reduction to a single equation and iterative procedure

In order to build the solution of the system (53)–(54), first we rewrite it as a single nonlinear integral equation

$$\varphi^2=\mathcal{K}\left[\frac{\left(-q^2\partial_t^2+1\right)\mathcal{K}\varphi}{\varphi}\right]. \tag{56}$$

This allows us to construct the following iterative procedure

$$\varphi_{n+1}(t)=-\varepsilon(t)\sqrt{\mathcal{K}\left[\frac{\mathcal{K}_q\varphi_n}{\varphi_n}\right]} \tag{57}$$

where integral operator $\mathcal{K}_q$ is defined by equation (40), and zero approximation is taken as

$$\varphi_0(t)=-\varepsilon(t) \tag{58}$$

where $\varepsilon(t)$ is a step function defined by equation (13). Please note that, although here we discuss the iterative procedure for $\varphi(t)$, we can also write a similar procedure for $\sigma(t)$ taking as a zero approximation a continuous function $\sigma_0(t)=1$. We can see that equation (57) assumes that $\varphi_n\neq 0$, which makes us define φ_0 at $t=0$ in a special way; see the following section for further details.

4.3. Numerical results

We have used the iterative procedure (57) to investigate solutions of the system (53)–(54).

As mentioned in the previous section, we used $-\varepsilon(t)$ as a zero iteration. Since in equation (57) we have to inverse φ_n to build the $(n+1)$ iteration, we define $\varepsilon(t)$ on the rhs of equations (57) and (58) to be non-zero for $t=0$. The basic idea here is to make the value of $\varepsilon(0)$ randomly[1] take values ± 1. In particular, we have used the following relation

$$\varepsilon(0)=(-1)^n \tag{59}$$

where n denotes the iteration number. We also tested some more complex random sequences, but the iterative procedure with equation (59) was the fastest and more predictable. We can also take ε at $t=0$ to be constant, for example $\varepsilon(0)=1$, but this led to some asymmetry in the resulting solutions.

The results of the iterative procedure (57) for $q=q_{\text{string}}$ are presented in figure 7. Note that the solution φ has a break at $t=0$.

[1] The author is grateful to L V Joukovskaya for the idea to use randomness in deterministic algorithms.

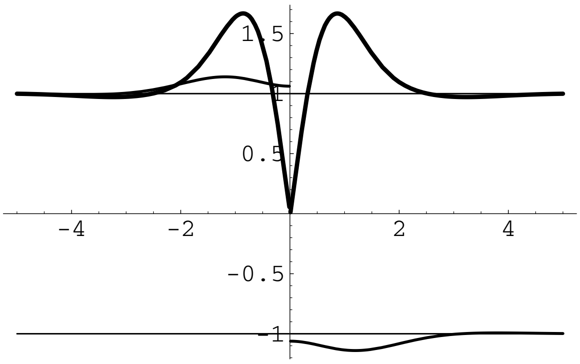

Figure 7. Results of iterative procedure solving the system (53)–(54): $\varphi(t)$ (thin line) and $\sigma(t)$ (thick line) for $q^2 = 0.96$.

It was found that there exists a critical value of the parameter q^2: $q_{\mathrm{cr}}^2 \approx 2.24$. This is the maximum value of q for which interpolating solutions exist. For $q^2 > q_{\mathrm{cr}}^2$ the iteration procedure starting from some step faces the negative argument under the square root in the rhs of equation (57). This means that the iterative procedure (57), while valid for finding interpolating solutions, fails to find oscillatory solutions.

4.4. Linearization of the system for large t

In this section we investigate the behaviour of the system (53)–(54) for $q \leqslant q_{\mathrm{cr}}$ in the large t limit. The basic idea is analogous to section 3.4. We represent the function φ which forms the solution for $q \neq 0$ as a deviation from φ_0, the solution for $q = 0$. We write a linear integral equation for this deviation in the large t limit.

Let us write the solution of (53)–(54) as a sum

$$\varphi(t) = \varphi_0(t) + \chi(t) \tag{60}$$

where $\varphi_0(t)$ is the solution for $q = 0$. Substituting equation (44) into equation (56) and leaving only linear terms in $\chi(t)$ we obtain

$$\mathcal{K}\frac{\mathcal{K}_q\varphi_0}{\varphi_0} + \mathcal{K}\frac{\mathcal{K}_q\chi}{\varphi_0} - \mathcal{K}\left(\chi\frac{\mathcal{K}_q\varphi_0}{\varphi_0^2}\right) = \varphi_0^2 + 2\varphi_0\chi. \tag{61}$$

Using the fact that φ_0 solves the system for $q = 0$ and $\varphi_0(t) \simeq 1$ and its derivatives are zero for large t we obtain

$$\mathcal{K}\mathcal{K}_q\chi - \mathcal{K}\chi = 2\chi. \tag{62}$$

Representing $\chi(t)$ as

$$\chi(t) = \mathrm{e}^{\mathrm{i}\Omega t}$$

and substituting it into equation (62) we obtain the following characteristic equation

$$(q^2\Omega^2 + 1)\,\mathrm{e}^{-\frac{1}{2}\Omega^2} - \mathrm{e}^{-\frac{1}{4}\Omega^2} = 2. \tag{63}$$

Analogously to what we did in section 3.4 we consider equation (63) as an equation for a complex variable Ω with parameter q^2. There is a minimum value of q^2 $\left(q_0^2 \approx 3.05\right)$ for which Ω is real. For $q^2 < q_0^2$ it has solutions with non-zero imaginary parts which give the oscillatory regime with exponentially decreasing amplitude. For $q^2 > q_0^2 \approx 3.05$ solutions are real. As mentioned in the previous section, numerical computations give the critical value $q_{\rm cr}^2 \approx 2.24$. This difference probably means that we cannot put φ_0 equal to 1 in the rhs of equation (62) or even consider nonlinear terms in χ in equation (61).

4.5. Asymptotic behaviour for large q

Here we investigate asymptotic behaviour of the system (53)–(54) for large q. As in section 3.5 we introduce a change of variables analogous to equation (49)

$$\chi(t) = \varphi(qt) \qquad \xi(t) = \sigma(qt).$$

In terms of $\chi(t)$ and $\xi(t)$ the system is rewritten as

$$(\mathcal{P}_q\xi)(t) = \chi(t)^2 \tag{64}$$

$$\left(-\partial_t^2 + 1\right)(\mathcal{P}_q\chi)(t) = \xi(t)\chi(t) \tag{65}$$

where integral operator $\mathcal{P}_q$ is defined by equation (51).

Performing computations analogous to section 3.5 we find that in the large q limit the system (64)–(65) becomes

$$\xi(t) = \chi(t)^2 \tag{66}$$

$$\left(-\partial_t^2 + 1\right)\chi(t) = \xi(t)\chi(t) \tag{67}$$

which is equivalent to an anharmonic oscillator.

5. Differential form of the equations and physical roots

In this section we provide physical details including actions for which the corresponding equations of motion form the subject of this paper.

Effective p-adic action is given by [2–4]

$$S = \frac{1}{g_p^2}\int \mathrm{d}^d x\left[-\frac{1}{2}\phi p^{-\frac{1}{2}\Box}\phi + \frac{1}{p+1}\phi^{p+1}\right] \qquad \frac{1}{g_p^2} \equiv \frac{1}{g^2}\frac{p^2}{p-1}. \tag{68}$$

Here ϕ is a scalar field which describes the tachyon in the p-adic string model, $x = (t, \vec{x})$ are d-dimensional space–time coordinates, p is a prime number, g_p is a coupling constant (g is universal coupling constant), the D'Alamber operator is defined in a standard way

$$\Box = -\frac{\partial^2}{\partial t^2} + \nabla\cdot\nabla \tag{69}$$

and operator $p^{-\frac{1}{2}\Box}$ is understood in the sense of expansion

$$p^{-\frac{1}{2}\Box} = \mathrm{e}^{-\frac{1}{2}\ln p\Box} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\ln p\right)^n\frac{1}{n!}\Box^n. \tag{70}$$

The corresponding equations of motion are

$$p^{-\frac{1}{2}\Box}\phi = \phi^p. \tag{71}$$

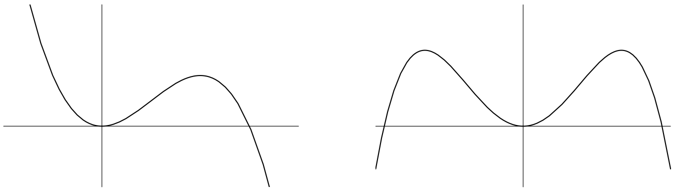

Figure 8. The p-adic potential for $p = 2$ (left) and $p = 3$ (right).

For homogeneous field configurations equation (71) is rewritten as follows

$$p^{\frac{1}{2}\partial_t^2}\phi = \phi^p. \tag{72}$$

If we consider slowly varying solutions and neglect high-order derivatives in the left-hand side (lhs) of equation (72) we obtain equations for an anharmonic oscillator

$$\tfrac{1}{2}\ln p\partial_t^2\phi + \phi = \phi^p \tag{73}$$

in the potential

$$V(\phi) = \frac{2}{\ln p}\left(\frac{1}{2}\phi^2 - \frac{1}{p+1}\phi^{p+1}\right). \tag{74}$$

From equation (74) we see that for the cases with even and odd p we have qualitatively different behaviours [6].

In particular, for $p = 2$ the potential (74) has a minimum when $\phi = 0$ (figure 8) and a maximum when $\phi = 1$. In [6] it was proven that there are no monotonic solutions interpolating between these points.

For $p = 3$ the potential has three extremal points:

$$\phi_0^{(1)} = 1 \qquad \phi_0^{(2)} = 0 \qquad \phi_0^{(3)} = -1. \tag{75}$$

The extremal points $\phi_0^{(1,3)}$ correspond to unstable vacua, and $\phi_0^{(2)}$ corresponds to the stable one. In the previous sections we were interested in the time-dependent solutions symmetrically interpolating between $\phi_0^{(1)}$ and $\phi_0^{(3)}$.

The action for the scalar field with the lowest mass square (tachyon) in the fermionic string model in the first non-trivial approximation is given by [9]

$$S[\upsilon, \psi] = \int \mathrm{d}t\left[\frac{1}{4}\upsilon(t)^2 + \frac{q^2}{2}(\partial_t\psi(t))^2 + \frac{1}{2}\psi^2(t) - \frac{1}{2}\Upsilon(t)\Psi^2(t)\right] \tag{76}$$

where

$$\Upsilon(t) = \exp\left(-\tfrac{1}{8}\partial^2\right)\upsilon(t) \qquad \Psi(t) = \exp\left(-\tfrac{1}{8}\partial^2\right)\psi(t) \tag{77}$$

and

$$q^2 = q^2_{\mathrm{string}} = -\frac{1}{4\ln\gamma} \approx 0.96.$$

Here, ψ is the tachyon field and υ is the auxiliary field. We write the equations of motion in terms of Υ and Ψ while the physical fields ψ and υ are obtained using the inverse of equation (77).

First let us consider an approximation for the action (76) which leads to simpler equations of motion. If we assume that we can neglect the smoothness of auxiliary field, i.e. in the

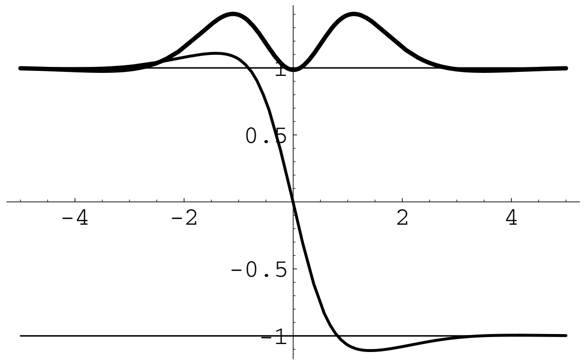

Figure 9. Physical fields $\psi(t)$ (thin line) and $\upsilon(t)$ (thick line) are smooth for all t ($q^2 = q^2_{\text{string}}$).

interacting term write υ instead of Υ, we obtain the following approximate action [9]

$$S[\upsilon, \psi] = \int \mathrm{d}t \left[\frac{1}{4}\upsilon(t)^2 + \frac{q^2}{2}(\partial_t\psi(t))^2 + \frac{1}{2}\psi^2(t) - \frac{1}{2}\upsilon(t)\Psi^2(t)\right]. \tag{78}$$

This action leads to the following equations of motion

$$\upsilon(t) = \Psi^2(t) \tag{79}$$

$$(-q^2\partial^2 + 1)\,\mathrm{e}^{\frac{1}{4}\partial_t^2}\Psi(t) = \upsilon(t)\Psi(t). \tag{80}$$

The first equation gives the expression for υ in terms of Ψ; substituting it to the second equation we obtain

$$(-q^2\partial^2 + 1)\,\mathrm{e}^{\frac{1}{4}\partial_t^2}\Psi(t) = \Psi(t)^3. \tag{81}$$

If we rewrite equation (81) in the integral form (see equation (3) with $a = 1/4$) we obtain

$$(-q^2\partial^2 + 1)\mathcal{K}\Psi = \Psi^3. \tag{82}$$

This equation was studied in section 3; see equation (37).

The equations of motion for original action (76) are

$$\mathrm{e}^{\frac{1}{4}\partial^2}\Upsilon(t) = \Psi^2(t) \tag{83}$$

$$(-q^2\partial^2 + 1)\,\mathrm{e}^{\frac{1}{4}\partial^2}\Psi(t) = \Upsilon(t)\Psi(t). \tag{84}$$

In the integral form this system is equivalent to the system (53)–(54) which was studied in section 4.

Let us compare the dynamics of physical field $\psi(t)$ obtained from the exact action (76) and approximate action (78). We compute the physical field $\psi(t)$ from solutions of equations (81) and (83)–(84) using the inverse of equation (77) which has the following form

$$\psi(t) = \mathrm{e}^{\frac{1}{8}\partial^2}\Psi(t) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-2(t-t')^2}\Psi(t')\,\mathrm{d}t'. \tag{85}$$

Note that, although Ψ solving equations (83)–(84) has a break at $t = 0$ (see section 4.3), the application of equation (85) makes the resulting physical field ψ smooth; see figure 9.

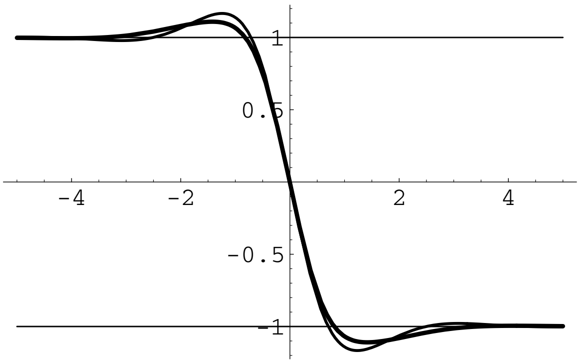

Figure 10. Comparison of physical field dynamics $\psi(t)$ obtained from exact action (76) (thick line) and approximate action (78) (thin line) for $q^2 = q^2_{\mathrm{string}}$.

The results of the comparison of physical fields obtained from exact and approximate actions are presented in figure 10.

6. Conclusions

In this paper we have numerically studied nonlinear equations depending on one variable with an infinite number of derivatives. Two different regimes for the solution of equations of motion for the fermionic string model are found and the critical value of the parameter is derived. It would be interesting to explore in more detail the transition between the two regimes. Also, a generalization of the results of the paper to the case of several variables would be important. It could be extremely interesting to generalize to the case describing the interaction of open and closed fermionic strings.

Acknowledgments

This work was partly supported by RFFI-02-01-01084 and RFFI-MAS-03-01-06466. The author is grateful to I Ya Aref'eva, V V Belokurov, D P Demichev, B Dragovich, L V Joukovskaya, A Yu Khrennikov and I V Volovich for fruitful discussions.

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